PRECISION OF PARABOLIC ELEMENTS

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Abstract

Formulae are given for a least-squares parabola on an O-C diagram, with calculation of the parabolic elements and their mean errors. The formulae are applied to TY Sct, an eleven-day classical Cepheid. The resulting rate of change of period is +0.000032 day per year ± 0.000016.

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1. Introduction

In an earlier issue of this Journal (Belserene 1989) I showed how to use the method of least squares to find the equation of a line through points on an O-C diagram, and then use the results to find the so-called linear elements of a variable star, along with their mean errors. The least-squares line gives us the best possible constant period for the plotted data. Many O-C diagrams, however, are not well approximated by straight lines.

The next most complicated case is a period changing at a constant rate. The resulting O-C diagram looks like a parabola. Our task will be to put the best possible parabola through the observed points. To find the equation of a line, we solved two equations for two unknowns. A parabola requires three. The recipe becomes considerably more complicated, but anyone who worked through the linear recipe might like to try the more difficult parabola on cases where the line is not very convincing.

As in the earlier paper, the steps are in numbered equations. The only unknown in each equation, by the time it is reached, is on the left hand side. The steps are to be executed in numerical order, except that a few equations, (25) through (28) are repeated with different values of some of their terms. The procedure is specifically for O-C analysis, but the central part of the process, equations (16) through (42), is suitable for any XY plot. In the equations, multiplication is indicated by an asterisk in the manner of computer languages.

The recipe for the line, which used only upper case letters without subscripts, required almost the whole alphabet. The recipe for the parabola needs to introduce subscripts on some of its symbols. I will, however, keep the same un-subscripted notation wherever possible and will use lowercase letters wherever the notation differs from the linear case. The subscripts make the recipe look more complicated, and the large number of steps makes the process tedious, but each step is a fairly small and straightforward bit of arithmetic. The recipe is intended for any patient person who is interested in results and willing to accept the formulae without proof. A knowledge of calculus would be needed to understand how the formulae had been derived, but they are presented here without derivations. No knowledge of calculus is needed to use them.

2. Astronomical Background

O-C analysis starts with knowing the approximate period and epoch of a variable star. If the period is not constant, we start with an approximation to the average period. (The epoch is a chosen time of maximum for a pulsating star, minimum for an eclipser. The text will say "maximum," letting the reader make the substitution if necessary.) Information about the true period is contained in the deviations, O-C, of observed times of maximum from times computed using the old epoch and period. If a plot of O-C against time (or cycle count) looks like a parabola, then the behavior of the star can reasonably be approximated by a period changing at a constant rate.

The parabolic elements of a variable star are c_0, c_1, and c_2 in an equation like

$$JD_{(max)} = c_0 + c_1*E + c_2*E^2.$$

We shall be able to calculate the c's as soon as we have worked out the equation of the parabola. An optional step in the method of least squares gives the mean errors of the calculated quantities. As each step in the recipe is introduced it will be applied to TY Sct, an eleven-day Cepheid. Ten values of O-C are available at the Maria Mitchell Observatory in unpublished work by Christian Bailey.

As in the linear case, the first steps in the recipe are to assemble the (old) linear elements (Oosterhoff 1943) and the observed times of maxima,

M = (old) epoch of maximum	= 28755.4	(1)
P = (old) period	= 11.05302	(2)
N = number of observed times of maximum	= 10	(3)
O = observed times of maxima		(4)

Bailey's 10 values of O for our sample star are in Table I, column 1. These and the epoch, M, are in the form JD-2400000. Before going any further, however, we are going to replace the (old) epoch by an equivalent one that is as close as possible to the average of the values of O. This step saves much trouble later on. Let

$\langle O \rangle$ = the average of the N values of O	= 33721.628	(5)
t = $\langle O \rangle$-M	= 4966.228	(6)
i = closest integer to t/P	= 449	(7)

Then we can define a revised (old) epoch as

m = M + P*i	= 33718.206	(8)

Next we need, for each observation, the number of whole cycles between m and O. It is usually the closest integer to the quotient (O-m)/P:

E = number of whole cycles between m and O (9)

The values of E for TY Sct are in Table I, column 2. With these values of E we compute the corresponding times of maximum according to the revised old elements:

C = computed times of maximum
= m + P*E (10)

and the differences between observation and computation:

Y = O - C (11)

The 10 values of C and Y for the sample star are in columns 3 and 4. The points in Figure 1, the O-C diagram, are the ten values of Y plotted against E.

If the observations and the adopted epoch and period were perfect, then all of the values of Y would be zero and the points would all lie exactly on the E axis. If the period had been constant but not equal to P, then all of the points would lie along a line inclined to the E axis. The slope of the line is the required correction to the period. In the case of TY Sct, however, the points suggest that a smooth curve might be a slightly better approximation than a straight line. The continuously changing slope of the curve implies a continuosly changing period for the star. The curve in Figure 1 is the least-squares parabola, calculated from the 10 pairs of values, Y and E, according to the steps which follow.

It is convenient to start by substituting for E a very slightly different quantity, X, defined by

$$D = \text{average of the N values of E} = +0.3 \qquad (12)$$
$$X = E - D \qquad (13)$$

The values of X are in column 5. This small change will turn out to be important when we come to step 20. An additional substitution is optional but helpful. In most O-C analysis, the values of X can get rather large, over 900 in the case of TY Sct. We are going to define another variable, lower-case x, which we will get by dividing upper-case X by a handy round number like 10 or 100 or 1000, chosen to make the largest quotient (without regard for sign) somewhere near 1, say between .3 and 3. For TY Sct I chose this scale factor, f, to be 1000 because the most extreme X value is -933.

$$f = \text{a handy round number} = 1000 \qquad (14)$$
$$x = X/f \qquad (15)$$

The values of x for TY Sct are in column 6 and along the top border of Figure 1.

Now it is time to leave the astronomy and see how to find the equation of the best parabola that can be drawn through scattered points. Later we shall return to astronomy to find the parabolic elements of TY Sct and their mean errors.

3. Least-Squares Parabolae

If the points on a graph of Y against x all lie on a parabola, then the values of Y are related to the corresponding values of x by an equation of the form $Y = a_0 + a_1*x + a_2*x^2$. It would be possible to take any three values of Y and the corresponding values of x, form three equations, and solve for the three unknowns (a_0, a_1, and a_2) by methods taught in algebra. The case under consideration, however, consists of more than three points. The points suggest a parabola but they scatter around it rather than falling right along it. No matter how we choose a_0, a_1, and a_2, we cannot make $a_0 + a_1*x + a_2*x^2$ come out exactly equal to the corresponding value of Y. With the best values for a_0, a_1, and a_2, the formula will give us not Y but some nearby value; let us call it y. Lower-case y defines a point exactly on a parabola, above or below the observed point, which lies at upper-case Y.

The method of least squares is designed to choose a_0, a_1, and a_2 to make the values of y come out as close to the values of Y as possible. The method is designed specifically for cases, like this one, where Y is subject to error while x is known exactly or almost so. How should we choose a_0, a_1, and a_2? Various authors (e.g., Chauvanet 1893, sections 28 and 29) give us three equations, the so-called "normal equations," that the a's must satisfy.

$$k_0*a_0 + k_1*a_1 + k_2*a_2 = u_0$$
$$k_1*a_0 + k_2*a_1 + k_3*a_2 = u_1$$
$$k_2*a_0 + k_3*a_1 + k_4*a_2 = u_2$$

The u's on the right and the k's on the left are related to the x's and the Y's by the rules in the next eight steps.

$$u_0 = \text{sum of the N values of Y} = 1.05 \qquad (16)$$
$$u_1 = \text{sum of the N values of x*Y} = -0.39323 \qquad (17)$$
$$u_2 = \text{sum of the N values of } x^2\text{*Y} = 0.76239 \qquad (18)$$

and

$$k_0 = N = 10 \qquad (19)$$
$$k_1 = \text{sum of the N values of x} = 0.0000 \qquad (20)$$
$$k_2 = \text{sum of the N values of } x^2 = 3.12599 \qquad (21)$$
$$k_3 = \text{sum of the N values of } x^3 = 0.16807 \qquad (22)$$
$$k_4 = \text{sum of the N values of } x^4 = 1.92143 \qquad (23)$$

Note that high powers of x appear, up to the fourth power. The values of the k's would have been unwieldy if we had tried to use uppercase X instead of taking the optional steps in equations (14) and (15).

Sometimes the value of k_1, the sum of the values of x, does not come out to be exactly zero, but that is only because of the accumulation of rounding errors in forming the values of X and x from the values of E; k_1 is really exactly zero from the way X and x are defined in equations (12) to (15). The terms with k_1 drop out and the normal equations become simpler:

$$\begin{aligned} k_0*a_0 \qquad\qquad &+ k_2*a_2 = u_0 \\ k_2*a_1 &+ k_3*a_2 = u_1 \\ k_2*a_0 + k_3*a_1 &+ k_4*a_2 = u_2 \end{aligned}$$

At this stage, the u's and the k's are known and we have three simultaneous equations for the unknowns, a_0, a_1, and a_2, but the unknowns are not yet on the left hand sides. There are several ways to solve three simultaneous equations for their three unknowns. All are tedious, but it helps a good deal that we have arranged to have k_1 equal to zero. The procedure adopted here in steps (24) through (28) is very much shorter than it might have been.

It starts by evaluating the denominator and numerator of a rather complicated formula for a_2:

$$\begin{aligned} d &= k_4 - k_2*k_2/N - k_3*k_3/k_2 \\ &= 1.92143 - 0.97718 - 0.00904 = 0.93521 \qquad (24) \end{aligned}$$

$$\begin{aligned} n &= u_2 - k_2*u_0/N - k_3*u_1/k_2 \\ &= 0.76239 - 0.32823 + 0.02114 = 0.45530 \qquad (25) \end{aligned}$$

The unknowns become known in the next three steps:

$$a_2 = n/d = 0.48684 \qquad (26)$$
$$\begin{aligned} a_1 &= (u_1 - a_2*k_3)/k_2 \\ &= (-0.39323 - 0.08182)/3.12599 = -0.15197 \qquad (27) \end{aligned}$$
$$\begin{aligned} a_0 &= (u_0 - a_2*k_2)/N \\ &= (1.05 - 1.52186)/10 = -0.04719 \qquad (28) \end{aligned}$$

Users are reminded that this method of solution works only because we used steps 12 and 13 to make $k_1 = 0$. Otherwise the formulae would have been more complicated.

Now we are in a position to see how close the parabola comes to the data points. Points on the parabola all satisfy the equation

$$y = a_0 + a_1*x + a_2*x^2 \qquad (29)$$

The ten values of y, corresponding to the ten values of x for TY Sct, are in Table I, column 7. I also used equation (29) to plot the parabola in Figure 1, calculating y for about forty equally spaced values of x.

The differences Y - y are called the residuals of the points from the line:

$$R = Y - y \tag{30}$$

These residuals and their squares are in columns 8 and 9. The sum of the squares of the residuals is the important quantity when it comes to assessing the overall quality of the parabola. This sum is as small as possible when the a's are chosen by the formulae of least squares, which is how the method got its name. The next steps in the recipe are to find this sum and a related quantity, which I call F in these two recipes.

$$Q = \text{sum of the N values of } R^2 = 0.39480 \tag{31}$$
$$F = Q/(N-3) = 0.05640 \tag{32}$$

Readers of the recipe for the line might notice that F had a slightly different definition there; the denominator was N-2. In each case the number of observations is decreased by the number of quantities that had been determined from them. The square root of F is what is called an unbiassed estimate of the errors of observation

$$\text{m.e. of an O-C} = \text{square root of F} = 0.237 \tag{33}$$

The size of the mean error for TY Sct, $\pm$0.237 day, is 2% of the cycle length. This is quite typical of the accuracy with which O-C can be determined by the methods we use with our photographic data at the Maria Mitchell Observatory. A much larger calculated mean error would have alerted us to two possibilities, either that the data were of lower than usual accuracy or that a parabola was a poor choice for the O-C diagram under consideration.

The mean error of an O-C is good to know, but it is not the only one we want. We want to know the precision of the three coefficients in the equation of the parabola. To find the mean errors of the a's, whose values we found in equations (26), (27), and (28), we must solve the simultaneous equations again, three times over, with different right-hand sides. First forget the results of equations (16) - (18). Instead, set

$$u_0 = 1, \quad u_1 = 0, \quad \text{and} \quad u_2 = 0 \tag{34}$$

Next repeat steps (25) through (28) with these substitutions until a new value of a_0 is found, ignoring a_1 and a_2. Then, with this value of a_0, not with the original one, calculate

$$g_0 = F*a_0 = 0.05640*0.20449 = 0.01153 \tag{35}$$

Then change the u's again, this time to

$$u_0 = 0, \quad u_1 = 1, \text{ and } \quad u_2 = 0 \tag{36}$$

and go through steps (25) through (27) to find a_1 (ignoring a_0 and a_2). Use this value of a_1, not the original one, to find

$$g_1 = F*a_1 = 0.05640*0.32299 = 0.01822 \tag{37}$$

Finally change the u's again, this time to

$$u_0 = 0, \quad u_1 = 0, \text{ and } \quad u_2 = 1 \tag{38}$$

and go through steps (25) and (26) to find the new value of a_2

(ignoring a_0 and a_1), to find

$$g_2 = F*a_2 = 0.05640*1.06928 = 0.06031 \quad (39)$$

The mean errors of a_0, a_1, and a_2 are

$$m_0 = \text{square root of } g_0 = 0.1074 \quad (40)$$
$$m_1 = \text{square root of } g_1 = 0.1350 \quad (41)$$
$$m_2 = \text{square root of } g_2 = 0.2456 \quad (42)$$

At last we have not only the numerical values of the coefficients in the equation for the parabola, equation (29), but also measures of their precision.

The recipe steps (16) through (42) are applicable to any least-squares parabola, not just an O-C diagram, provided that the errors are in Y but not in X, and provided further that the value of k_1 has been set to zero as this recipe did when it defined X in equations (12) and (13). Several more steps are needed to produce the astronomical results which were the motivation for the calculation.

4. The New Elements and Their Errors

When the elements are parabolic, the formula for the JD's of maximum has three terms:

$$JD_{(max)} = c_0 + c_1*E + c_2*E^2.$$

It can be shown that the improved elements of the variable star are

$$c_2 = a_2/f^2 = 0.000000487 \quad (43)$$
$$c_1 = \text{New period} = P + a_1/f - 2*c_2*D = 11.0528677 \quad (44)$$
$$c_0 = \text{New epoch} = m + a_0 - a_1*D/f + c_2*D^2 = 33718.1589 \quad (45)$$

The mean errors of the c's are, almost exactly

$$e_2 = m_2/f^2 = 0.0000002456 \quad (46)$$
$$e_1 = m_1/f = 0.0001350 \quad (47)$$
$$e_0 = m_0 = 0.1074 \quad (48)$$

Equations (46) to (48) would be exact if D were exactly zero. This recipe, however, has made sure that D is quite small so the approximation is almost exact. Often the terms with c_2*D and c_2*D^2 in equations (44) and (45) are also negligible.

The astronomical result of all this calculation is, then, after a little more rounding,

$$JD_{(max)} = JD\ 2433718.16 + 11.05287\ E + 0.00000049\ E^2.$$
$$\pm 0.11 \quad \pm 0.00014 \quad \pm 0.00000025$$

Another astronomical result, interesting to astrophysicists studying stellar evolution, is the rate of change of the period. Expressed in days per year the rate and its mean error are

$$\text{rate} = 2 * c_2 * 365.25/P = +0.000032 \text{ day/year.} \quad (47)$$
$$\text{m.e.} = 2 * e_2 * 365.25/P = \pm 0.000016 \quad (48)$$

The mean errors were a nuisance to calculate, but they are worth the trouble since they show us how much uncertainty is introduced in the parabolic elements by the scatter in the observations. The coefficient of the E^2 term is particularly interesting. For TY Sct it is not quite twice as large as its mean error. Error theory claims that observed quantities will be affected by random errors this large or larger about five percent of the time, so we have some confidence that the third term is meaningful. The coefficients, however, are not directly observed quantities. A more rigorous assessment of the reality of the third term can be made with an F-test (Pringle 1975) which uses statistical theory in a comparison of residuals from the parabola with residuals from a line. The F-test result for TY Sct (not a part of this recipe) is less supportive of the reality of the curvature. It gives a 9% chance that the apparent curvature is due to random errors in the values of O-C. A statistician would ask us to be wary.

For some stars the value of e_2 and/or the F-test give considerable confidence in the reality of the curvature. In the case of TY Sct it would not be difficult to believe that the apparent curvature is simply due to uncertainties affecting the 10 points. Bailey included data from 1918 to 1979 in his ten O-C points. It will soon be time for someone to look at the more recent plates.

5. Summary

A method has been given, without derivation, for the calculation of parabolic elements through least squares analysis of O-C data, with emphasis on the determination of the mean errors of the new elements. The steps are in the numbered equations (1) through (48) in the order in which they are to be carried out, except that equations 25 through 28 are each used more than once, with different values for some of their terms. The central part of the procedure, equations (16) through (42), applies to least-squares parabolae in general. Computer programs using this method are in use at the Maria Mitchell Observatory in our studies of periodic variable stars. Our variable-star research receives support from National Science Foundation grant AST86-19885. Chris Bailey's work on TY Sct received funding from AST78-07405.

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TABLE I

Least-Squares Calculation of Parabolic Elements for TY Sct

O	E	C	Y	X	x	y	R	R*R
23406.07	-933	23405.74	0.33	-933.3	-0.9333	0.5187	-0.1887	0.03561
27606.33	-553	27605.89	0.44	-553.3	-0.5533	0.1859	0.2541	0.06457
29098.10	-418	29098.04	0.06	-418.3	-0.4183	0.1016	-0.0416	0.00173
29905.27	-345	29904.91	0.36	-345.3	-0.3453	0.0633	0.2967	0.08803
32402.79	-119	32402.90	-0.11	-119.3	-0.1193	-0.0221	-0.0879	0.00773
34502.62	71	34502.97	-0.35	70.7	0.0707	-0.0555	-0.2945	0.08673
35398.26	152	35398.27	-0.01	151.7	0.1517	-0.0590	0.0490	0.00240
39498.72	523	39498.94	-0.22	522.7	0.5227	0.0064	-0.2264	0.05126
41599.34	713	41599.01	0.33	712.7	0.7127	0.0918	0.2382	0.05674
43798.78	912	43798.56	0.22	911.7	0.9117	0.2189	0.0011	0.00000

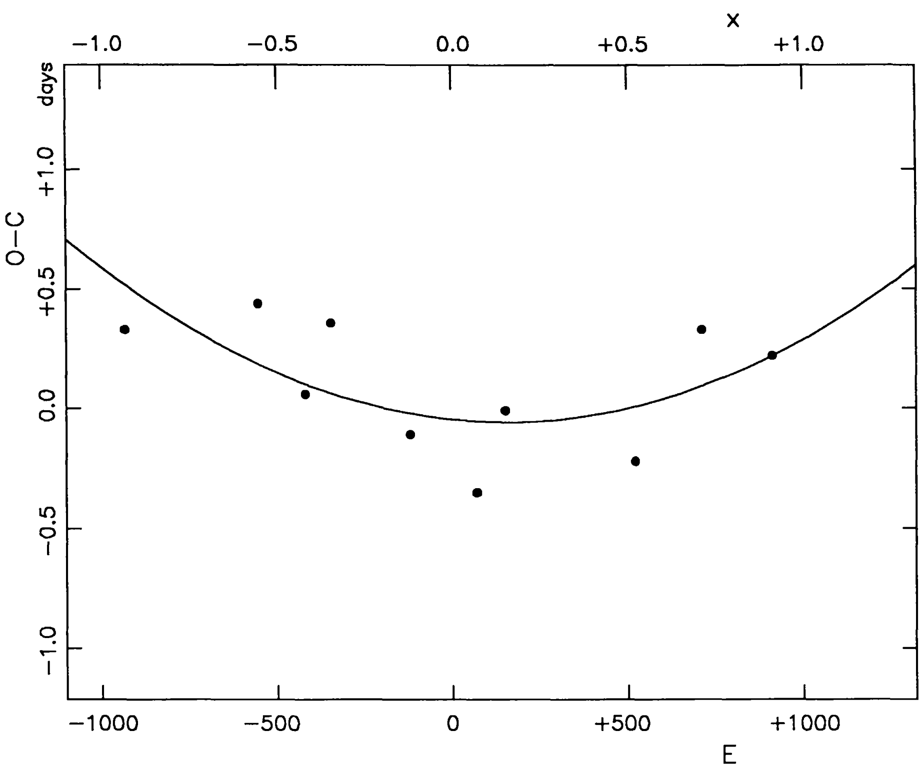

Figure 1. O-C diagram for TY Sct. The curve is the least-squares parabola, equation (29), with its parameters calculated from equations (16) through (28).